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Dimensional Crossover, the Renormalization Group and Finite Size Scaling. [1]

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Abstract: In this paper we review some of our recent investigations into dimensional crossover using renormalization group arguments. We consider field theory on $E = R^d \times B$ where B is a compact space of "size" L. By employing an L dependent renormalization we obtain renormalization group equations that interpolate between the two limits $L = \infty$ and $L = 0$, the latter being equivalent to the absence of B. We consider two distinct cases: one where the interpolation is between two renormalizable theories and another where it is between a non-renormalizable theory and a renormalizable one. In the former we find that the correlation functions depend only on the scaling variable $\frac{L}{\xi_L}$ where ξ_L is the correlation length on E. This is finite size scaling. The scaling variable becomes $tL^{\frac{1}{\nu}}$ when $\frac{L}{\xi_L} \to \infty$, $\xi_L \to \infty$, and $tL^{\frac{1}{\nu'}}$ when $\xi_L \to \infty$ for fixed L. Here ν and ν' are the correlation length exponents of the $L = \infty$ and $L = 0$ systems respectively and t is the mass parameter on E. The results of this paper have many applications in statistical mechanics and particle physics, e.g. liquid helium in wafer geometries, finite temperature field theory and Kaluza Klein theories.

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1 Introduction

Many problems of interest in theoretical and experimental physics involve a large (possibly infinite) number of degrees of freedom. When many degrees of freedom play a role one would expect field theoretic techniques to be applicable. For instance in lattice systems, where the intuition that has been built up over the last few decades, especially in the application of field theoretic techniques to critical phenomena, indicates that a field theoretic approximation is applicable when the correlation length (Compton wavelength of field fluctuations) is several lattice spacings. In other words field theory should be applicable once the wavelengths of interest are large comparable to any microscopic cutoff.

A particular case of interest occurs when we consider a lattice which is infinite in some directions but finite in others. If the finite directions are large compared to the lattice spacing one expects a field theory description to be appropriate where one describes the underlying lattice as a continuum but with the global geometry of the lattice. It is this continuum viewpoint that we wish to explore but with the lattice picture in the background as our ultraviolet (UV) regulator, though for technical reasons we will actually use dimensional regularization. The global geometry we will consider is $E = R^d \times B$ where R^d represents d dimensional Euclidean space and B is a compact manifold. In the present discussion we will restrict our considerations to manifolds which are of this product type.

This contribution is intended as a review of some of the work we have been doing in applying field theory, and in particular the field theoretic renormalization group, to describe the crossover that occurs when one varies the correlation length ($\xi_L = m_L^{-1}$) of the field fluctuations relative to the size of B, which we will call L. Intutively one feels that the effect of the compact dimensions should drop out of the problem as the Compton wavelength becomes large compared to L, i.e. as $m_L L \to 0$, in a similar way to which the original lattice disappeared from the problem. Thus one expects a genuine crossover from the physics of one dimension (called the bulk dimension, the bulk system being the system obtained when $L \to \infty$) to that of the lower dimension d. The question we wish to address is how this can be described quantitatively using field theory. More generally this problem falls under the heading of "finite size effects"and we will investigate an interesting scaling that arises as the size is scaled relative to m_L called "finite size scaling"[1, 2].

One of the powerful mathematical tools used in theoretical physics is the renormalization group (RG). It has been developed in two seperate settings along rather different lines. In the lattice setting it corresponds to a thinning out of the lattice degrees of freedom by either a lattice decimation procedure or an averaging prescription wherein a group of lattice degrees of freedom are averaged over to give a new effective degree of freedom. From this point of view one easily sees that in the case of a lattice version of the geometry E the RG changes from that of the bulk (when the averaging occurs at scales very small compared to L) to one associated with R^d as one continues to decimate (or average) to larger and larger scales. Eventually all trace of B disappears and one has a truly lower dimensional RG. [4] This contrasts rather severely with the point of view that has been popular in field theory which closely associates the RG with the UV infinities of the theory and its subsequent independence of the scale entering in the regularization process which is necessitated by these infinities. The UV infinities of field theory are well known to be associated with microscopic structure and are characterized by the nature and symmetries of the field and the dimension of the space at the microscopic level. We are led to the conclusion that either an association of the RG with the UV infinities is inappropriate being in conflict with the lattice picture or that the two RGs are essentially unrelated. We find the former conclusion to be in fact valid and will in the following demonstrate that the association of the field theoretic RG with purely UV infinities is misleading. Instead we find a RG that incoprorates the crossover mentioned above in a natural way. The coefficents arising in this renormalization group are functions of κL where κ loosely speaking is our coarse graining scale.

In field theory the RG equation arises as an expression of the fact that the bare theory is independent of the renormalization point at which we have chosen to represent the physical amplitudes of the theory. If Z_ϕ is the wave function renormalization constant, the renormalized and bare N-point functions are related by $\Gamma^N_R = Z_\phi^{N/2}\Gamma^N_B$. For a renormalizable theory we will typically have in addition to wave function renormalization, mass renormalization and coupling constant renormalization. The RG equation we get then depends on how we choose our counterterms, which is the field theoretic version of how we define our averaged or decimated degrees of freedom.

In the context of perturbation theory a very convenient choice of counterterm is obtained by what is called minimal subtraction. In this scheme the counterterms take on a particularly simple form, in that they are independent of the mass. If we consider this scheme for the manifold E we know that the UV divergences in such a "box" are purely local (since these arise from the very short distance fluctuations of the theory), consequently the form of the counterterms arising in this scheme are the same for the finite system as for the bulk system. The resulting RG will be the same as that of the bulk and we will subsequently never see a crossover. Now, although one can eliminate all the UV divergencies using the bulk counterterms, one is not restricted to minimal subtraction, or choosing one's counterterms to be just those of the bulk system. One can in fact absorb any amount one wishes of the finite contributions to the diagrams into the counterterms. This will change the RG equation one obtains, and a judicious choice may allow one to extract more information from the theory.

In fact from the above arguments it is apparent, as was shown in [3], that it is essential to use an L dependent subtraction scheme if one wishes to recover perturba-

[4]The resulting theory on R^d can of course depend on the global properties of B.

tively the dimensionally reduced system, which arises in the limit $m_L L \to 0$, without encountering new divergences. These divergences are not of an UV nature but are associated with the fact that the effective coupling constant becomes large leading to a complete breakdown in perturbation theory. The point is that an L independent subtraction scheme implies an expansion about the bulk infrared fixed point, however, for fixed L one knows that if there is a fixed point associated with the finite system it is this one that will be relevant. In other words the new divergences are arising because one is expanding around the wrong fixed point. An expansion around the correct reduced fixed point can be achieved by implementing an appropriate L dependent scheme such as normalization conditions at fixed L, as we will use in this paper, or by choosing a non minimal subtraction that includes all terms that diverge as $\kappa L \to 0$ or $\kappa L \to \infty$, the two limits we wish to consider. The latter will give counterterms that are L dependent but still mass independent, thereby preserving most of the advantages of the minimal prescription. The resulting β-functions and anomalous scaling dimensions are explicitly L dependent.

Now let us present some of the general features of our approach in the context of a particular Euclidean field theory, that of a scalar field. We consider an action of the form

$$S = \int_E d^d x d^{d'} y [\frac{1}{2}(\frac{\partial \phi_B(x,y}{\partial x})^2 + (\frac{\partial \phi_B(x,y)}{\partial y})^2 - \frac{1}{2} m_B^2 \phi_B^2) + L_{int}], \tag{1}$$

where x and y are co-ordinates on R^d and B respectively. Our approach is to Fourier expand the field $\phi(x,y)$ in the compact dimensions[5], $\phi(x,y) = \sum_N \phi_N(x) Y_N(y)$, where $Y_N(y)$ are eigenfunctions of the Laplace operator on B. Substituting into the action and integrating over y we obtain

$$S = -\sum_N \int_{R^d} d^4 x \frac{1}{2} \phi_N(x)(\partial^2 + m^2 + M_N^2)\phi_N(x) + S_{int} \tag{2}$$

where $m^2 + M_N^2$ appear as d-dimensional masses, M_N^2 being the eigenvalues of the Laplacian on B. when $M_N^2 = 0$ we have massless modes in the lower dimension at the tree level, the remaining modes have M_N^2 proportional to L^{-2}. If $L^2 m^2$ is small then the latter modes correspond to an infinite tower of heavy particles. Now if the energy scales of our probes are less than the threshold energy for the creation of these heavy particles they will never be seen in our experiments. These modes do however, make a contribution to the correlation functions through loop corrections. That this contribution is finite or small is not obvious due to their infinite number. If this number were finite then we would have the decoupling theorems [4] to reassure us, however in the case of an infinite number no such theorem has been proven, thus the first complication we encounter is whether decoupling is valid or not.

The second complication is to be seen in the nature of the infinite tower of modes refelecting the fact that the theory is a higher dimensional one in disguise, which may

[5]This is only a technical device for clarity of discussion and not essential to the results.

not be renormalizable even though the lower dimensional one is. In general therefore the nature of the UV divergencies must change because of the infinite summation over massive modes. In the language of phase transitions the relevant phenomenon is the evolution from one critical behaviour to another. When the infinite tower of massive modes yields a non-renormalizable interaction the bulk system is said to be above the upper critical dimension. When it yields a renormalizable or super renormalizable theory we are dealing with crossover at or below the uppercritical dimension. In this setting we expect infrared divergences to be very important, as we must obtain from the above classical action non negligible anomalous dimensions. It is in this setting that the RG is in its element, having been applied at its inception to the problem of the "infrared catastrophe"in QED [5].

The structure of the paper is as follows. In section 2 we examine what we term the renormalizable to renormalizable problem for the special case of $R^{3-\epsilon} \times S^1$. In section 3 the non-renormalizable to renormalizable problem is treated for the example of $R^{4-\epsilon} \times S^1 \times S^1$ and Section 4 contains a derivation of finite size scaling in a general renormalizable to renormalizable problem and some concluding remarks.

2 Renormalizable to Renormalizable crossover

In this section we consider crossover on the manifold $R^{3-\epsilon} \times S^1$, the reader is referred to [3] for a more detailed treatment of the material in this section. In this case the interaction Lagrangian in (1) is taken to be $L_{int} = -\frac{\lambda_B}{4!}\phi_B^4(x,y)$ where we use the subscript B to label bare quantities. Substituting the Fourier expansion into the action (1) and integrating out the extra coordinate we get the dimensionally reduced action on $R^{3-\epsilon}$ the quadratic part of which is given by (2) with $M_n^2 = (\frac{2\pi n}{L})^2$ where n labels the Fourier modes on the S^1 forming the space B. This yields an interaction Lagrangian from the lower dimensional point of view $L_{int} = -\frac{u_B \Lambda^{1+\epsilon}}{4!}\phi_{B0}^4(x) + L'_{int}$, where L'_{int} includes heavy modes with $n^2 \neq 0$. The coupling constant u_B is related to the four dimensional one by $u_B \Lambda = \frac{\lambda_B}{L}$. We have extracted the scale Λ, which is some scale associated with the bare theory, so as to work with dimensionless couplings, since only dimensionless couplings can be large or small in and of themselves. Throughout we will work only with such couplings.

We consider for simplicity and purposes of illustration only the one loop four-point function with external legs corresponding to light modes with $n = 0$. Only N-point functions with the lowest mode on the external legs are relevant if the energies of the external particles are much smaller than L^{-1}. This in reality is not a restriction though since all modes couple with the same strength as is apparent if one takes the higher dimensional point of view. All renormalizations are then strictly related to how the lowest mode renormalization is done. Γ^4 evaluated at the symmetric point

$p_i p_j = \frac{p^2}{4}(4\delta_{ij} - 1)$, where p_i are incoming momenta, to one loop is given by

$$\Gamma^4(p, L, m_B, u_B) = u_B \Lambda^{1+\epsilon} - u_B^2 \Lambda^{2+2\epsilon} p^{-(1+\epsilon)} F(Lp, \frac{m}{p}), \tag{3}$$

where

$$F(pL, \frac{m}{p}) = \frac{3}{2} \frac{\Gamma(\frac{1+\epsilon}{2})}{(4\pi)^{2-(\frac{1+\epsilon}{2})}} \sum_{n=-\infty}^{\infty} \int_0^1 \frac{dt}{\left(t(1-t) + (\frac{2\pi N}{Lp})^2 + (\frac{m}{p})^2\right)^{\frac{1+\epsilon}{2}}}. \tag{4}$$

t being a Feynman parameter introduced into the evaluation of the diagram. The function $F(pL, m/p)$ given by (4) is divergent when ϵ goes to zero due to the infinite summation (the sum can be understood in the sense of ζ-function regularization). We can make this explicit by examining the series representation for $F(pL, \frac{m}{p})$ in the limit of small pL

$$\begin{aligned} F(pL, \tfrac{m}{p}) = \; & F_0(\tfrac{m}{p}) \\ & + \tfrac{3}{2(4\pi)^{2-\frac{1+\epsilon}{2}}} \textstyle\sum_{k=0}^{\infty} \sum_{l=0}^{k} (-)^k \tfrac{\Gamma(k-l+1)^2 \Gamma(\frac{1+\epsilon}{2}+k) \zeta(\frac{1+\epsilon}{2}+k)}{\Gamma(k+1)\Gamma(2k-2l+2)} (\tfrac{m}{p})^{2l} \left[(\tfrac{pL}{2\pi})^2\right]^{\frac{1+\epsilon}{2}+k} \end{aligned} \tag{5}$$

where $F_0(\frac{m}{\kappa})$ is the $n^2 = 0$ term in (5) (the three dimensional result), and $\zeta(\nu) = \sum_{n^2 \neq 0} (n^2)^{-\nu}$ is the generalized ζ-function. In the present example $\zeta(\nu) = 2\zeta_R(2\nu)$, ζ_R being the Riemann zeta function. The Riemann zeta function has a pole when $2\nu = 1$ which in the above indicates that the $k = 0$ term is singular as $\epsilon \to 0$. The pole term is extracted from $\frac{3}{(4\pi)^{2-\frac{1+\epsilon}{2}}} \Gamma(\frac{1+\epsilon}{2}) \zeta_R(1+\epsilon)(\frac{pL}{2\pi})^{1+\epsilon}$ as $\epsilon \to 0$. If we were to perform minimal subtraction the pole is the only term we would subtract, however, there is also a $lnLp$ term which is infrared divergent. Thus minimal subtraction only picks up the four dimensional UV divergences and is incapable of handling the three dimensional infrared divergences, it is therefore not sufficient for our purposes. Let us concentrate on the special case $m^2 = 0$ in which case (5) simplifies to

$$F(pL) = F_0 + \frac{3}{(4\pi)^{2-(\frac{1+\epsilon}{2})}} \sum_{k=0}^{\infty} (-)^k \frac{\Gamma(k+1)}{\Gamma(2k+2)} \Gamma(\frac{1+\epsilon}{2} + k) \zeta_R(1 + \epsilon + 2k) (\frac{pL}{2\pi})^{1+\epsilon+2k} \tag{6}$$

F_0 being a numerical constant (finite when $\epsilon \to 0$ exhibiting the superrenormalizable nature of the three dimensional theory) given by

$$F_0 = \frac{3}{2} \frac{\Gamma(\frac{1+\epsilon}{2})}{(4\pi)^{2-(\frac{1+\epsilon}{2})}} \frac{\Gamma(\frac{1-\epsilon}{2})^2}{\Gamma(1-\epsilon)}. \tag{7}$$

We are now ready to discuss the renormalization prescription we advocate which will give the desired crossover behaviour. We will define corresponding dimensionless renormalized coupling constants by the following normalization conditions:

$$\Gamma^4|_{p^2=\kappa^2} = u\kappa^{1+\epsilon} \tag{8}$$

Implementing these conditions we obtain relations between the bare and the renormalized couplings. These relations are nonlinear and should be interpreted with care. In the current case they yield

$$u\kappa^{1+\epsilon} = u_B\Lambda^{1+\epsilon} - u_B^2\Lambda^{2+2\epsilon}\kappa^{-1-\epsilon}F(\kappa L), \tag{9}$$

Inverting the expansions (9) to obtain the bare coupling constants in terms of the renormalized ones and substituting into eq. (3) we get

$$\Gamma^4 = u\kappa^{1+\epsilon} - u^2\kappa^{1+\epsilon}\left[(\tfrac{\kappa}{p})^{1+\epsilon}F(Lp) - F(\kappa L)\right]. \tag{10}$$

In the above it is apparent that what we mean by u_B is $u(\Lambda)$ and the last equation should really be interpreted as being applicable when Λ is close to κ, since the relation between the two couplings is nonlinear. This leads us into the RG where we look at the infinitesimal change in κ keeping Λ fixed. The generator of this differential change is termed the β-function and defined as $\kappa\frac{\partial u}{\partial \kappa}$ at fixed Λ. We find it convenient for further use to introduce the operator $A_\nu = \kappa\frac{\partial}{\partial\kappa} - 2\nu$. Note the following useful identities $[A_{\nu_1}, A_{\nu_2}] = 0$ and $\kappa^{2\nu}A_0\kappa^{-2\nu} = A_\nu$. The β-function is then $\beta(u,\kappa L) = A_0 u$ and from (9) we obtain

$$\beta(u) = -(1+\epsilon)u + u^2 f(\kappa L) \tag{11}$$

where we have defined $f(\kappa L) = -A_{\frac{1+\epsilon}{2}}F(\kappa L)$. $\beta(u)$ interpolates in the desired fashion between the three dimensional and four dimensional cases as κL is varied between zero and infinity respectively.

An alternative approximation to this function can be obtained by expanding the expression for $F(\kappa L)$ in $1+\epsilon = \epsilon'$ and assuming this quantity is small. Then one retains in the counterterms the pole term in ϵ' together with all terms that diverge in either limit $\kappa L \to 0$ or $\kappa L \to \infty$. This corresponds to the usual ϵ expansion as applied to three dimensional problems, $(L = 0)$ and because of its similarity with minimal subtraction is termed Generalized Minimal Subtraction. This procedure yields for the problem under consideration the β function

$$\beta(u,\kappa L) = -(1+\epsilon)u + \frac{3u^2}{16\pi^2}\frac{\kappa L}{2}\coth(\frac{\kappa L}{2}) \tag{12}$$

The crossover features of both expressions are the same [3]. One important point to note is that the coupling constant as we approach the bulk system is no longer u but $\lambda = u\kappa L$, the four dimensional coupling. One finds it has a β-function

$$\beta(\lambda) = -\epsilon\lambda + \frac{3\lambda^2}{16\pi^2}\coth\frac{\kappa L}{2}$$

As $\kappa L \to 0$, u has a fixed point $u^* = \frac{16\pi^2(1+\epsilon)}{3}$ which is associated with the reduced system, and in the limit $\kappa L \to \infty$ one recovers the "bulk" fixed point $\lambda^* = \frac{32\pi^2\epsilon}{3}$. We will postpone further discussion of this renormalizable case till section 4 where we give a general treatment of the features in a somewhat formal mannner.

3 Non-renormalizable to renormalizable crossover.

We now wish to consider the dimensional reduction of a nonrenormalizable theory to a renormalizable one. For this purpose we will consider the dimension d to be $4-\epsilon$ and the manifold B to be the two-dimensional torus $T^2 = S^1 \times S^1$ with the radii of the both circles $L/2\pi$ (this is purely for convenience); ϵ can be viewed as the regularization parameter in dimensional regularization. Because the theory is now non-renormalizable many additional parameters enter its renormalization. In our approach we assume that all new couplings that the theory requires due to non-renormalizability are of the order of the power of the basic ϕ^4 coupling appearing in the first divergent diagram which contributes to their renormalization. This allows us to control the proliferation of divergences which now grows with the loop order. The details of the computations for this section are contained in [6].

The action is now given by (1) where the interaction Lagrangian is (using dimensionless couplings)

$$L_{int} = -\frac{\hat{\lambda}_{1B}\Lambda^{-2+\epsilon}}{4!}\phi_B^4(x,y) - \frac{\hat{\lambda}_{2B}\Lambda^{-4+\epsilon}}{4!}\phi_B(x,y)\Box\phi_B^3(x,y) - \frac{\hat{\lambda}_{3B}\Lambda^{-6+2\epsilon}}{6!}\phi_B^6(x,y),$$

and $\Box$ is the D'Alambertian on E. We have introduced the second and third terms in L_{int} on the bare level since counterterms with such structures are necessary for subtracting one-loop UV divergencies in this theory. The eigenvalues of the Laplacian on B in this case are $M_N^2 = (\frac{2\pi N}{L})^2$ where $N^2 = n_1^2 + n_2^2$ and the intergers n_1 and n_2 label the Fourier modes on the S^1's and the interaction Lagrangian becomes

$$L_{int} = -\frac{\lambda_{1B}\Lambda^{\epsilon}}{4!}\phi_{B0}^4(x) - \frac{\lambda_{2B}\Lambda^{-2+\epsilon}}{4!}\phi_{B0}(x)\Box_{(4)}\phi_{B0}^3(x) - \frac{\lambda_{3B}\Lambda^{-2+2\epsilon}}{6!}\phi_{B0}^6(x) + L'_{int}, \quad (13)$$

where L'_{int} includes heavy modes with $N^2 \neq 0$, $\Box_{(4)}$ is the D'Alambertian on $R^{4-\epsilon}$ and the coupling constants $\lambda_{iB}, (i = 1,2,3)$ are related to the original "multidimensional"ones by $\lambda_{1B} = \frac{\hat{\lambda}_{1B}}{\Lambda^2 L^2}$, $\lambda_{2B} = \frac{\hat{\lambda}_{2B}}{\Lambda^2 L^2}$, and $\lambda_{3B} = \frac{\hat{\lambda}_{3B}}{\Lambda^4 L^4}$. Again Λ, is some scale associated with the bare theory.

As in the previous section we consider the four-point function with light modes ($N^2 = 0$) on the external legs. Only such processes are relevant when the energies of the external particles are much smaller than L^{-1}. In many Kaluza Klein compactification schemes [7] L turns out to be of the order of the Planck length $L_{planck} \approx 10^{-33}cm$. Calculating the standard one loop diagram proportional to λ_{1B}^2 for the four-point vertex function Γ^4 evaluated at the symmetric point $p_ip_j = \frac{p^2}{4}(4\delta_{ij} - 1)$, where p_i are incoming momenta and t a Feynman parameter we obtain,

$$\Gamma^4(p, L, m_B, \lambda_{1B}, \lambda_{2B}) = \lambda_{1B}\Lambda^{\epsilon} + p^2\lambda_{2B}\Lambda^{-2+\epsilon} - \lambda_{1B}^2\Lambda^{2\epsilon}p^{-\epsilon}I(pL, \frac{m}{p}), \quad (14)$$

where

$$I(Lp, \frac{m}{p}) = \frac{3}{2}\frac{\Gamma(\frac{\epsilon}{2})}{(4\pi)^{2-\frac{\epsilon}{2}}} \sum_{n_1,n_2=-\infty}^{\infty} \int_0^1 \frac{dt}{(t(1-t) + (\frac{2\pi N}{Lp})^2 + (\frac{m}{p})^2)^{\frac{\epsilon}{2}}}. \quad (15)$$

We assume that $\lambda_{2B} \simeq \lambda_{1B}^2$ so that the one-loop diagrams proportional to $\lambda_{1B}\lambda_{2B}$ and λ_{2B}^2 can be neglected. Note that if we do not make this assumption the proliferation of divergences becomes uncontrollable. For a more detailed discussion of this point see [6].

Let us now discuss the renormalization prescription. Again the sum can be understood in the sense of ζ-function regularization and if one performs a Taylor expansion as in the previous section, one finds that the pole term arises from the limit $\epsilon \to 0$ of

$$I_{div}(pL, \frac{m}{p}) = \frac{3}{2}\frac{\Gamma(\frac{\epsilon}{2})}{(4\pi)^{2-\frac{\epsilon}{2}}}\left[(\frac{p}{m})^{\epsilon} + (\zeta(\frac{\epsilon}{2}) - \frac{\epsilon}{2}\zeta(1+\frac{\epsilon}{2}))((\frac{p}{m})^{2} + \frac{1}{6})(\frac{pL}{2\pi})^{2+\epsilon}\right] \tag{16}$$

where the generalized ζ-function appearing in the above expressions is given by $\zeta(\nu) = \sum_{N^2\neq 0}(N^2)^{-\nu}$ and has a pole at $\nu = 1$. A minimal subtraction prescription would retain only the pole part of (16) in the counterterms, however a prescription similar to that applied in the previous section is preferable here also. We see that the function $I(pL, \frac{m}{p})$ given by (15) and its first derivatives with respect to p^2 and m^2 are divergent when ϵ goes to zero, reflecting the six-dimensional character of the original theory. Thus there are two undetermined parameters which correspond to the renormalizations of the operators ϕ_B^4 and $\phi_B \Box_{(4)} \phi_B^3$ in the bare Lagrangian. Our normalization conditions for the corresponding dimensionless renormalized coupling constants are:

$$\begin{aligned} \frac{\partial \Gamma^4}{\partial p^2}\Big|_{p^2=\kappa^2} &= \lambda_2 \kappa^{-2+\epsilon} \\ (1 - p^2\frac{\partial}{\partial p^2})\Gamma^4\Big|_{p^2=\kappa^2} &= \lambda_1 \kappa^{\epsilon} \end{aligned} \tag{17}$$

which yield the following relationships between the bare and renormalized couplings

$$\begin{aligned} \lambda_1\kappa^{\epsilon} &= \lambda_{1B}\Lambda^{\epsilon} + \lambda_{1B}^2\Lambda^{2\epsilon}\kappa^{-\epsilon}\frac{1}{2}A_{\frac{\epsilon}{2}+1}I(\kappa L, \tfrac{m}{\kappa}), \\ \lambda_2\kappa^{-2+\epsilon} &= \lambda_{2B}\Lambda^{-2+\epsilon} - \lambda_{1B}^2\Lambda^{2\epsilon}\kappa^{-2-\epsilon}\frac{1}{2}A_{\frac{\epsilon}{2}}I(\kappa L, \tfrac{m}{\kappa}), \end{aligned} \tag{18}$$

using the operator A_ν defined in the previous section. Then β-functions for these couplings are

$$\begin{aligned} \beta(\lambda_1) &= -\epsilon\lambda_1 + \lambda_1^2 J(\kappa L) \\ \beta(\lambda_2) &= (2-\epsilon)\lambda_2 - \lambda_1^2 J(\kappa L) \end{aligned} \tag{19}$$

where we found it convenient to define a function $J(\kappa L) = \frac{1}{2}A_{\frac{\epsilon}{2}+1}A_{\frac{\epsilon}{2}}I(\kappa L)$ which is a well behaved function when $\epsilon \to 0$. $J(0) = a_2 = \frac{3\Gamma(2+\frac{\epsilon}{2})}{(4\pi)^{2-\frac{\epsilon}{2}}}\frac{\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)}$ and is the usual four dimensional coefficient.

Analysing the relations (18) in the limit of small κL we see that the coupling constant λ_2 still has a finite renormalization. Since in this limit the internal space disappears we would expect to recover standard four-dimensional formulae where λ_2 is not renormalized. That the coupling constant λ_2 remains as a running coupling in this limit may be troubling at first sight, however, by defining a slightly different set of couplings as follows $g_1 = \frac{\lambda_1}{1+\frac{\epsilon}{2}} + \frac{1}{2}\left(\frac{\lambda_1}{1+\frac{\epsilon}{2}}\right)^2 J(\kappa L)$, $g_2 = \frac{\lambda_2}{1+\frac{\epsilon}{2}} - \frac{1}{2}\left(\frac{\lambda_1}{1+\frac{\epsilon}{2}}\right)^2 J(\kappa L)$ the coupling g_2 can be shown to be unrenormalized in this limit. The $\frac{1}{1+\frac{\epsilon}{2}}$ arises as $\frac{2}{d_2-2d_1}$ where d_1 and d_2 are the powers of κ with which the couplings λ_1 and λ_2 enter into Γ^4, in this case $-\epsilon$ and $2-\epsilon$ respectively. The β-functions for these couplings are then

$$\begin{aligned} \beta(g_1) &= -\epsilon g_1 + g_1^2 \tilde{J}(\kappa L) \\ \beta(g_2) &= (2-\epsilon) g_2 - g_1^2 S(\kappa L) \end{aligned} \tag{20}$$

where for convenience we have defined $S(\kappa L) = \frac{1}{4} A_0 A_{\frac{\epsilon}{2}} A_{1+\frac{\epsilon}{2}} I(\kappa L) = \frac{1}{2} A_0 J(\kappa L)$ and $\tilde{J}(\kappa L) = J(\kappa L) + S(\kappa L)$. Using series expansions one can demonstrate that these RG equations have the property that when $\kappa L \to 0$ the coupling constant g_2 is not renormalized, since in this limit $S(\kappa L) \to 0$. This is a desirable feature since this coupling is not necessary in the four dimensional theory as it does not undergo an infinite renormalization. Similarly, in this limit the renormalization of g_1 reduces to the four dimensional result, since a_2 is all that survives and is exactly the result obtained by doing the calculation purely in four dimensions. That this is true is essentially a demonstration of decoupling of the infinite tower of massive modes.

Two other couplings related to the above are possibly the most illuminating since in the dimensional crossover these couplings do not vary significantly, they are $h_1 = \lambda_1 J(\kappa L)$ and $h_2 = \lambda_2 J(\kappa L)$. These have beta functions

$$\begin{aligned} \beta(h_1) &= -\epsilon(\kappa L) h_1 + h_1^2 \\ \beta(h_2) &= (2-\epsilon(\kappa L)) h_2 - h_1^2 \end{aligned} \tag{21}$$

where $\epsilon(\kappa L)$ is a function that interpolates between ϵ and $\epsilon+1$. More explicitly $\epsilon(\kappa L) = \epsilon - \frac{1}{J(\kappa L)} \kappa \partial_\kappa J(\kappa L)$. The corresponding set of couplings to (g_1, g_2) and the coupling u of the previous section can be defined in a similar fashion. If in the above we had used minimal subtraction instead of L dependent normalization conditions we would find that the finite contribution to Γ^4 become infinite in the limit $\kappa L \to 0$. Before turning to the solution of these RG equations we note that in the large κL limit the functions $J(\kappa L)$, $\tilde{J}(\kappa L)$ and $S(\kappa L)$ are proportional to $(\kappa L)^2$ plus exponentially small terms. These functions are diverging in this limit because the volume of the internal manifold was absorbed into the couplings when we did the Fourier transform and this volume is now diverging. Thus the couplings (g_1, g_2) and (λ_1, λ_2) which are natural for the four dimensional limit $(\kappa L \to 0)$ are inappropriate for the six dimensional one $(\kappa L \to \infty)$. We define six-dimensional couplings $\bar{g}_i = (\kappa L)^2 g_i$ and equivalently for the set (λ_1, λ_2). The coupling constants (h_1, h_2) naturally incorporate the above re-definitions in these limits. The RG equations in this limit in terms of $(\bar{g}_1, \bar{g}_2)$ have the form

$$\beta(\bar{g}_1) = (2-\epsilon)\bar{g}_1 + \bar{g}_1^2 \tilde{J}(\kappa L)$$
$$\beta(\bar{g}_2) = (4-\epsilon)\bar{g}_2 - \bar{g}_1^2 S(\kappa L) \qquad (22)$$

where as $\kappa L \to \infty$, $\tilde{J}(\kappa L) \to 2b$, $S(\kappa L) \to b$ where b is a constant. These are the natural six dimensional RG equations for this system.

Since we have performed a renormalization of all terms necessary to make the results finite in the six-dimensional case and recover the four-dimensional RG equations in the $\kappa L \to 0$ limit, we have RG equations that interpolate, in what we believe to be a natural way, between the four and six dimensional theories. The non-triviality of the equation for the additional coupling g_2 as $\kappa L \to \infty$ reflects the non-renormalizability of the six-dimensional theory. In general we expect the same features to persist in higher loop calculations, however a proliferation of additional parameters will arise in our prescription. The essential feature of our work is that it brings into the realm of calculability the corrections due to additional dimensions, even when working with non renormalizable theories.

The RG equations (21) can be solved without difficulty to obtain

$$h_1(\kappa) = \frac{h_1(\kappa_0) exp[-\int_{\kappa_0}^{\kappa} \epsilon(xL)\frac{dx}{x}]}{1 - h_1 \int_{\kappa_0}^{\kappa} \frac{dy}{y} exp[-\int_{\kappa_0}^{y} \epsilon(xL)\frac{dx}{x}]} \qquad (23)$$

$$h_2(\kappa) = exp[\int_{\kappa_0}^{\kappa}(2-\epsilon(xL))\frac{dx}{x}][h_2(\kappa_0) - \int_{\kappa_0}^{\kappa} h_1(y)^2 exp[-\int_{\kappa_0}^{y}(2-\epsilon(xL))\frac{dx}{x}]\frac{dy}{y}] \qquad (24)$$

In the above κ_0 is an initial renormalization point, and the solution tells us how the coupling changes as the renormalization point is changed. The solutions of the equations (20) are obtained by substituting back for the original variables. We obtain

$$\lambda_1(\rho) = \frac{\lambda_1(1)\rho^{-\epsilon}}{1 - \lambda_1(1)\int_1^{\rho} dx x^{-\epsilon-1} J(x\kappa_0 L)} \qquad (25)$$

$$\lambda_2(\rho) = \rho^{2-\epsilon}[\lambda_2(1) - \int_1^{\rho} dx \lambda_1^2(x) x^{-3+\epsilon} J(x\kappa_0 L)] \qquad (26)$$

where we have defined $\rho = \frac{\kappa}{\kappa_0}$. By direct analogy the solutions for the couplings $(h_1^{'}, h_2^{'})$ can be obtained yielding (g_1, g_2) to be

$$g_1(\rho) = \frac{g_1(1)\rho^{-\epsilon}}{1 - g_1(1)\int_1^{\rho} dx x^{-\epsilon-1} \tilde{J}(x\kappa_0 L)} \qquad (27)$$

$$g_2(\rho) = \rho^{2-\epsilon}[g_2(1) - \int_1^{\rho} dx g_1^2(x) x^{-3+\epsilon} S(x\kappa_0 L)] \qquad (28)$$

Since in the four-dimensional limit the coupling g_2 does not get renormalized it seems natural to choose the normalization condition such that $g_2 = 0$ in which case the theory reduces exactly to the four-dimensional one. This of course is a form of fine tuning in six dimensions, however it is preserved by the renormalization group flow and is natural from the four dimensional point of view. Our initial assumption of imposing the relationship $g_2 \sim g_1^2$ is similar to the fine tuning of Coleman and Weinberg in the case of scalar electrodynamics [8]. Note that the non-renormalizability of the theory begins to become important when we begin to probe the theory at scales of order L. In the limit $\kappa L \to 0$ $\bar{J}(\kappa L) = a_2$ and (27) gives a Landau pole at $\rho = \rho^*$. If $g_2(1) = 0$ or is fine tuned to be small then $g_2(\rho)$ remains small relative to g_1 (and our assumption is valid) for $1 \leq \rho << \rho^*(\kappa L)$; otherwise our assumption is not self-consistent and other diagrams must be considered.

4 Finite size scaling

The renormalization prescriptions discussed in the previous sections are chosen to have explicit dependence on L by requiring that all quantities divergent as $m_L L \to 0$ or ∞ are included as well as UV divergences. This means that the RG equation for a general N-point function takes the form

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(u,\kappa L)\frac{\partial}{\partial u} + \gamma_{\phi^2}(u,\kappa L)t\frac{\partial}{\partial t} - \frac{N}{2}\gamma_\phi(u,\kappa L)\right)\Gamma^{(N)}(k_i, L, u, t, \kappa) = 0 \quad (29)$$

where the Wilson functions $\gamma_{\phi^2} = -\kappa\frac{\partial}{\partial\kappa}\ln Z_{\phi^2}$, $\gamma_\phi = \kappa\frac{\partial}{\partial\kappa}\ln Z_\phi$, $\beta(u,\kappa L) = \kappa\frac{\partial}{\partial\kappa}u$ are dependent now on the coupling constant u and also explicitly on κL, in contrast to the bulk situation where they only depend on the bulk coupling constant λ. They are such that as κL ranges from 0 to ∞ they interpolate between the functions of the lower dimension and bulk. Here we adopt the notation t for the renormalized mass parameter since in critical phenomena it corresponds to $T - T_c(L)$ the temperature difference from the critical temperature at finite L as opposed to $T - T_c(\infty)$ which would correspond to the renormalized mass parameter of the bulk system. It is essential to use t if one is to get a sensible dimensionally reduced limit. In this section we examine the general consequences of having a RG equation of this type for the renormalizable to renormalizable problem. For further details of the material in this section the reader is refered to [2].

Equation (29) can be solved in the standard manner by the method of characteristics. Defining $t(\kappa_0) = t$, $\frac{\kappa}{\kappa_0} = \rho$ and solving the characteristic equations for t and Z_ϕ we obtain

$$t(\rho) = t\ \exp\left(\int_1^\rho \gamma_{\phi^2}(u(x, x\kappa_0 L), x\kappa_0 L)\frac{dx}{x}\right)$$

$$Z_\phi(\rho) = Z_\phi \exp[\int_1^\rho \gamma_\phi(u(x), xL)\frac{dx}{x}]$$

Similarly solving the β-function equation gives us $\lambda(\rho)$. For simplicity we suppress the L dependence of γ_ϕ, γ_{ϕ^2} and λ, and drop the subscript on κ_0. The solution to (29) is then

$$\Gamma^N(k_i, L, u, t, \kappa) = \exp\left(\frac{N}{2}\int_\rho^1 \gamma_\phi(x)\frac{dx}{x}\right)\Gamma^N(k_i, L, u(\rho), t(\rho), \rho\kappa) \tag{30}$$

where $u(1) = u$, and ρ is arbitrary, i.e. the right hand side of equation (30) is in fact independent of ρ as a consequence of RG invariance. Using dimensional analysis one can extract the dependence on $\rho\kappa$ as the overall dimensionful quantity and re-express (30) as

$$\Gamma^N(k_i, L, \lambda, t, u) = (\kappa\rho)^{d-N\frac{(d-2)}{2}}\exp\left(\frac{N}{2}\int_\rho^1 \frac{dx}{x}\gamma_\phi(x)\right)\Gamma^N(\frac{k_i}{\rho\kappa}, \rho\kappa L, \frac{t(\rho)}{\rho^2\kappa^2}, u(\rho), 1) \tag{31}$$

Since ρ is arbitrary we eliminate it by choosing $\frac{t(\rho)}{\rho^2\kappa^2} = 1$. This gives us an expression for $\rho = \rho(\frac{t}{\kappa^2}, \kappa L)$. Γ^N now depends on $\rho\kappa L$, the running coupling constant and $\frac{k_i}{\rho\kappa}$ if we choose not to set the external momenta to zero. This choice for ρ also enables one to evaluate the right hand side of (31) away from the critical regime and hence perturbatively. Note that using this condition we can substitute $t(\rho)$ for $\rho^2\kappa^2$ thus $\rho\kappa L = t(\rho)^{\frac{1}{2}}L$, but as defined $t(\rho) = \xi_L^{-2}$ [9], where ξ_L is the correlation length of the system of size L. The relevant scaling variable is therefore $\frac{L}{\xi_L}$, rather than $\frac{L}{\xi_\infty}$. Note that only as $m_L L \to \infty$ will these coincide. We see that $t(\rho) = \rho^2\kappa^2$ is equivalent to choosing $\rho\kappa\xi_L = 1$, rather than $\rho\kappa L = 1$ which was the choice of Brézin [10] in a proof of finite size scaling when there is no lower dimensional critical point. Our choice has the advantage of allowing us to probe the regime of divergent correlation length ($\xi_L \to \infty$) for finite L. Substituting $\rho = \frac{1}{\kappa\xi_L}$ into (31) we obtain the most general scaling form of Γ^N to be

$$\Gamma^N(k_i, L, t, u, \kappa) = (\xi_L)^{N(\frac{d-2}{2})-d}\exp\left(\frac{N}{2}\int_{\frac{1}{\kappa\xi_L}}^1 \gamma_\phi(x)\frac{dx}{x}\right)\Gamma^N(k_i\xi_L, \frac{L}{\xi_L}, u(\frac{1}{\kappa\xi_L})). \tag{32}$$

For a RG invariant such as $R^N = (\frac{\Gamma^N}{\Gamma})/(\frac{\Gamma^2}{\Gamma})^{\frac{N}{2}}$ one finds [2] that $R^N = f^N(\frac{L}{\xi_L})$. That one gets $\frac{L}{\xi_L}$ as the relevant scaling variable can be understood from another point of view in the case of periodic boundary conditions. Since the dependence on L enters through the fact that the momenta in the periodic direction now take discrete values quantized as $k_n = (\frac{2\pi n}{L})$, the dependence of Γ^N on the momenta is of the form $k\xi_L$ which gives $\frac{L}{\xi_L}$ as the expected dependence on L.

Let us now examine the neighbourhood of a fixed point. Specializing to the case $E = S^1 \times R^{3-\epsilon}$ there are two potential fixed points. It is important to specify which one is in question and this of course depends on the ratio of L to ξ_L. For fixed L there is only one true fixed point, the reduced one, however, for $\frac{L}{\xi_L} \to \infty$ with $\xi_L \to \infty$, the bulk fixed point emerges. For the moment we will not specify which fixed point is in question but treat the neighbourhood of an arbitrary one. In this case γ_ϕ, γ_{ϕ^2} and u approach their fixed point values, which we denote by affixing the superscript *. It is therefore convenient to expand around these values using what are termed "metric factors "to accommodate the fact that one is not exactly at the critical point. These metric factors are slowly varying away from the fixed point, unless one approaches another whereupon they diverge. They express the deviations from exact scaling and play the crucial role of taking us from one fixed point to another. We express $t(\rho) = t\rho^{\gamma^*_{\phi^2}} C_{\phi^2}$ where $C_{\phi^2} = \exp\left(\int_1^\rho (\gamma_{\phi^2}(x) - \gamma^*_{\phi^2})\frac{dx}{x}\right)$. Similarly $Z_\phi(\rho) = Z_\phi \rho^{\gamma^*_\phi} C_\phi$ where $C_\phi = \exp\left(\int_1^\rho (\gamma_\phi(x) - \gamma^*_\phi)\frac{dx}{x}\right)$. C_{ϕ^2} and C_ϕ are slowly varying metric factors near the fixed point $u = u^*$. Equation (31) therefore becomes

$$\Gamma^N(k_i, L, t, u, \kappa) = (\kappa\rho)^{d-N(\frac{d-2}{2})} \rho^{-\frac{N}{2}\gamma^*_\phi} C_\phi^{-\frac{N}{2}} \Gamma^N(\frac{k_i}{\rho\kappa}, \rho\kappa L, 1, u(\rho), 1) \tag{33}$$

where ρ is determined by $t(\rho) = \rho^2\kappa^2$ and now depends on the one metric factor. One subtlety is that in the neighbourhood of the lower dimensional fixed point the dimensions of the fields and the free energy density are different than at the bulk, this implies that $\Gamma^N = (\rho\kappa)^{-\frac{N}{2}(d-d')+(d-d')}\Gamma'^N$ where the prime is used to denote the lower dimensional quantity. With this identification we see that d will correctly become d' the dimension of the reduced system when we consider the reduced fixed point.

To obtain the dependence of ρ on t and L we need to examine the equation for the running temperature in more detail. In the neighbourhood of the fixed point we find it is of the form $\frac{t}{\kappa^2}\rho^{-2+\gamma^*_{\phi^2}} C_{\phi^2} = 1$. Now by definition $\gamma^*_{\phi^2} = 2 - \frac{1}{\nu^*}$, where ν^* is the correlation length exponent associated with the fixed point under consideration, therefore $\rho = (\frac{t}{\kappa^2})^{\nu^*} C_2$ where C_2 is a new L dependent metric factor (obtained from $(C_{\phi^2})^{\nu^*}$), which caters for the crossover. We emphasize that it is only near a fixed point that we get a scaling variable of the form Lt^{ν^*}, more generally it is $\frac{L}{\xi_L}$. A useful way of parameterizing the crossover is via an effective critical exponent $\nu_{eff} = -\frac{d\ln\xi_L}{d\ln t}$. We can then write $\frac{L}{\xi_L} = Le^{\int_\kappa^t \nu_{eff}\frac{dx}{x}}$. In the limits $\frac{L}{\xi_L} \to \infty$ or $\frac{L}{\xi_L} \to 0$ ν_{eff}, is t independent and becomes ν or ν', the bulk or reduced exponent respectively. Substituting back into (32), noting $\gamma^*_\phi = \eta^*$, we obtain

$$\Gamma^N(k_i, L, t, u, \kappa) = t^{\nu^*(d-\frac{N}{2}(d-2+\eta^*))} C_\phi^{-\frac{N}{2}} \Gamma^N(k_i t^{-\nu^*} C_2^{-1}, Lt^{\nu^*} C_2, 1, u(t^{\nu^*} C_2), 1) \tag{34}$$

for the N-point function in the limit as one of the fixed points is approached, where ν^* and η^* are the associated exponents. It is important to realize that the metric factors can be calculated within the formalism presented.

Sufficently near the bulk fixed point, for the metric factors to be regarded as equal to one, (34) can be rewritten as

$$\Gamma^N = \Gamma^N_\infty f(k_i t^{-\nu}, Lt^\nu) \tag{35}$$

where $\Gamma_\infty^N \sim t^{\nu(d-\frac{N}{2}(d-2+\eta))}$, which is the usual form of the scaling relation. If instead we are sufficently near the reduced fixed point we find

$$\Gamma^N = \Gamma_\infty'^N f'(k_i t^{-\nu'}, L t^{\nu'}) \tag{36}$$

where $\Gamma_\infty'^N \sim t^{\nu'(d'-\frac{N}{2}(d'-2+\eta'))}$ is the critical N-point function for the reduced theory, and f' is a finite size scaling function as seen from this perspective.

In this contribution we have reviewed our recent investigations into dimensional crossover. Although for pedagogical reasons we have concentrated on two particular crossovers — on $S^1 \times R^3$ and $S^1 \times S^1 \times R^4$ — we hope that it is clear that the formalism is general. One of the key points here is that the existence of finite dimensions can change the universality class of a field theory, i.e. to induce a crossover with $\frac{1}{L}$ acting as a relevant perturbation. What we have achieved here is the development of a formalism that can account for the changes induced by this relevant perturbation both qualitatively and quantitatively. In the introduction our intuition as to what should happen during the crossover was motivated by considering lattice renormalizations such as blocking or decimation. By exhibiting the crossover explicitly using continuum field theory methods we feel we have drawn the two formulations a little closer.

So, what are the possible applications of this formalism? Besides critical phenomena in real laboratory systems, e.g. liquid helium between two surfaces [11], it is fairly clear that both Kaluza-Klein theories and finite temperature quantum field theory [6, 12] are accessible. Cosmological phase transitions would be another interesting area. All the above fit rather well into the notion of having some finite dimension. As shown in [13] however, there are many systems that exhibit dimensional reduction in situations where the "finite" size nature is less obvious. A good example of this is an abelian Higgs model in a uniform magnetic field [14]. In this case the "finite" size effect arises due to the "confining" nature of the magnetic field on the Higgs particles, i.e semiclassically they are induced to move on Landau orbits. Crossover phenomenon in this case could be investigated using the present formalism, the role of $\frac{1}{L}$.

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